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Soliton equations and the zero curvature condition in noncommutative geometry

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Abstract. Familiar nonlinear and in particular soliton equations arise as zero curvature conditions for $GL(1,\mathbb{R})$ connections with noncommutative differential calculi. The Burgers equation is formulated in this way and the Cole–Hopf transformation for it attains the interpretation of a transformation of the connection to a pure gauge in this mathematical framework. The KdV, modified KdV equation and the Miura transformation are obtained jointly in a similar setting and a rather straightforward generalization leads to the KP and a modified KP equation. Furthermore, a differential calculus associated with the Boussinesq equation is derived from the KP calculus.

1. Introduction

Soliton equations are known to admit zero curvature formulations (see [1], for example). In case of the Korteweg–de Vries (KdV), sine-Gordon and sinh-Gordon equations, one can find $SL(2,\mathbb{R})$-connection 1-forms (gauge potentials) A such that the condition of vanishing curvature (or 'field strength')

$$F := \mathrm{d}A + AA = 0 \tag{1}$$

is equivalent to the respective soliton equation [2]. In this work we show that the Burgers, KdV, Kadomtsev–Petviashvili (KP) and Boussinesq equation can even be expressed as a zero curvature condition for a $GL(1,\mathbb{R})$-connection, but with respect to a *noncommutative* differential calculus. By the latter we mean an analogue of the calculus of differential forms on a manifold, but here functions and 1-forms in general do not commute. As a consequence, the product of a 1-form with itself need not vanish, in contrast to the case of the ordinary differential calculus. Because of this fact, one already obtains nontrivial equations from $F = 0$ for a single 1-form A (and not just for a matrix of 1-forms).

The relevant mathematical framework underlying this work is the theory of differential calculi on commutative algebras. An exposition to it can be found in [3], see also the references therein. In 'noncommutative geometry' an associative and not necessarily commutative algebra replaces the algebra of (smooth) functions on a manifold. A differential calculus on the algebra is then regarded as the most basic geometric structure on which further geometric concepts like connections can be defined. Though in this paper we still deal with commutative algebras and thus topological spaces, nontrivial commutation relations are introduced between functions and (generalized) 1-forms and this already catches much of the flair of general noncommutative geometry.

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Irrespective of the choice of a differential calculus, the expression (1) makes sense. A gauge transformation with an invertible function ψ is given by

$$A' = \psi A \psi^{-1} - \mathrm{d}\psi\, \psi^{-1} \tag{2}$$

and induces the transformation $F' = \psi F \psi^{-1}$ of the curvature 2-form†.

In section 2 the Burgers equation is obtained via a zero curvature condition with respect to a simple deformation of the ordinary differential calculus, and the Cole–Hopf transformation appears as a transformation to a pure gauge. In section 3 we start with two differential operators which play a role in the theory of the KdV equation. We construct a differential calculus in which these operators appear as generalized partial derivatives. From the zero curvature condition for a $GL(1,\mathbb{R})$ connection we then recover the KdV, modified KdV (mKdV) equation and the Miura transformation. Even more interesting is the fact that our treatment of the KdV equation naturally leads to the KP equation via a dimensional continuation of the differential calculus associated with the KdV equation. This is the subject of section 4. In section 5 we show how the Boussinesq equation and its associated differential calculus arises via a dimensional reduction of the calculus associated with the KP equation. section 6 contains some conclusions.

2. The Burgers equation

In the following, $\mathcal{A}$ denotes the algebra of C^∞-functions on $\mathbb{R}^2$, and t and x are the canonical coordinate functions on $\mathbb{R}^2$. Let $\Omega(\mathcal{A})$ be the differential calculus determined by

$$[\mathrm{d}t, t] = [\mathrm{d}x, t] = [\mathrm{d}t, x] = 0 \qquad [\mathrm{d}x, x] = \eta\, \mathrm{d}t \tag{3}$$

with a constant η. More generally, we have

$$\mathrm{d}t\, f = f\, \mathrm{d}t \qquad \mathrm{d}x\, f = f\, \mathrm{d}x + \eta f_x\, \mathrm{d}t \tag{4}$$

for $f \in \mathcal{A}$. Here f_x denotes the partial derivative with respect to x. For the differential of a function one obtains

$$\mathrm{d}f = \left(f_t + \frac{\eta}{2} f_{xx}\right) \mathrm{d}t + f_x\, \mathrm{d}x \tag{5}$$

and

$$\mathrm{d}x\, \mathrm{d}x = 0 = \mathrm{d}t\, \mathrm{d}t \qquad \mathrm{d}x\, \mathrm{d}t = -\mathrm{d}t\, \mathrm{d}x. \tag{6}$$

This calculus has already been explored in several papers, see [4] in particular. The differentials $\mathrm{d}t$ and $\mathrm{d}x$ constitute a basis of the space of 1-forms $\Omega^1(\mathcal{A})$ as a left or right $\mathcal{A}$-module.

A $GL(1,\mathbb{R})$-connection 1-form can be written as

$$A = w\, \mathrm{d}t + u\, \mathrm{d}x \tag{7}$$

with functions u, w. Using the differential calculus introduced above, the curvature becomes

$$F = \left(u_t + \frac{\eta}{2} u_{xx} + \eta u u_x - w_x\right) \mathrm{d}t\, \mathrm{d}x. \tag{8}$$

For $w = 0$ the zero curvature condition takes the form

$$u_t + \frac{\eta}{2} u_{xx} + \frac{\eta}{2} (u^2)_x = 0 \tag{9}$$

† More generally, this holds for any G-connection where G is a matrix group in which ψ has values.

which is the Burgers equation [5, 8]. On the other hand, it is easily verified that the zero curvature condition implies that A can be written as a 'pure gauge', i.e.

$$A = \theta^{-1}\,\mathrm{d}\theta = \theta^{-1}\left(\left[\theta_t + \frac{\eta}{2}\theta_{xx}\right]\mathrm{d}t + \theta_x\,\mathrm{d}x\right) \tag{10}$$

with an invertible function θ. Comparing the last expression with $A = u\,\mathrm{d}x$, we obtain

$$u = \theta_x/\theta \qquad \theta_t + \frac{\eta}{2}\theta_{xx} = 0. \tag{11}$$

Here we rediscover the Cole–Hopf transformation [8] which reduces the Burgers equation to the linear diffusion equation.

3. The KdV equation

A starting point in a modern treatment of the KdV equation is given by the two 'undressed' (cf [8]) differential operators

$$\Delta_1^{(0)} := \partial_t + ab\partial_x^3 \qquad \Delta_2^{(0)} := b\partial_x^2 \tag{12}$$

with nonvanishing real constants a and b. Let us try to understand these as generalized partial derivatives of a (noncommutative) differential calculus on $\mathcal{A} = C^\infty(\mathbb{R}^2)$. The associated exterior derivative should then act on a function $f \in \mathcal{A}$ as follows,

$$\mathrm{d}f = (f_t + abf_{xxx})\,\mathrm{d}t + bf_{xx}\xi + f_x\,\mathrm{d}x \tag{13}$$

where ξ is a 1-form which together with $\mathrm{d}t$ and $\mathrm{d}x$ constitutes a basis of the space of 1-forms as a left $\mathcal{A}$-module. The operator d has to satisfy the Leibniz rule

$$\mathrm{d}(fh) = (\mathrm{d}f)h + f\,\mathrm{d}h \tag{14}$$

for $f, h \in \mathcal{A}$. It leads to

$$(f_t + abf_{xxx})[h, \mathrm{d}t] + bf_{xx}([h, \xi] + 3ah_x\,\mathrm{d}t) + f_x([h, \mathrm{d}x] + 3abh_{xx}\,\mathrm{d}t + 2bh_x\xi) = 0. \tag{15}$$

This is only satisfied (for all smooth functions f, h) if the following commutation relations between functions and 1-forms hold,

$$\mathrm{d}t\,f = f\,\mathrm{d}t \qquad \xi f = f\xi + 3af_x\,\mathrm{d}t \qquad \mathrm{d}x\,f = f\,\mathrm{d}x + 2bf_x\xi + 3abf_{xx}\,\mathrm{d}t. \tag{16}$$

In particular,

$$[\mathrm{d}t, t] = [\mathrm{d}x, t] = [\mathrm{d}t, x] = [\xi, t] = 0 \qquad [\mathrm{d}x, x] = 2b\xi \qquad [\xi, x] = 3a\,\mathrm{d}t. \tag{17}$$

Furthermore, we have the 2-form relations

$$\mathrm{d}t\,\mathrm{d}t = \mathrm{d}t\,\mathrm{d}x + \mathrm{d}x\,\mathrm{d}t = \xi\xi = \xi\,\mathrm{d}t + \mathrm{d}t\,\xi = \xi\,\mathrm{d}x + \mathrm{d}x\,\xi = 0 \qquad \mathrm{d}\xi = -\frac{1}{b}\,\mathrm{d}x\,\mathrm{d}x. \tag{18}$$

They are obtained by acting with d on the equations (17) and by commuting x through the resulting relations using (17).

A connection 1-form can be written as

$$A = w\,\mathrm{d}t + u\xi + v\,\mathrm{d}x \tag{19}$$

with functions u, v, w. The curvature of A is

$$\begin{aligned} F = {} & [-b\,w_{xx} - 2bvw_x + u_t + abu_{xxx} + 3auu_x + 3abvu_{xx}]\,\mathrm{d}t\,\xi \\ & +[-w_x + v_t + abv_{xxx} + 3abvv_{xx} + 3auv_x]\,\mathrm{d}t\,\mathrm{d}x + (-u + bv_x + bv^2)_x\xi\,\mathrm{d}x \\ & + \left(-\frac{1}{b}u + v_x + v^2\right)\mathrm{d}x\,\mathrm{d}x. \end{aligned} \tag{20}$$

If $d\xi \neq 0$, the set of 2-forms $dt\,\xi$, $dt\,dx$, $dt\,dx$, $\xi\,dx$, $dx\,dx$ span the space $\Omega^2(\mathcal{A})$ of 2-forms as a left (and right) $\mathcal{A}$-module and will be assumed to be a left $\mathcal{A}$-module basis†. The zero curvature condition now becomes

$$-\frac{1}{b}u + v_x + v^2 = 0 \tag{21}$$

$$v_t + abv_{xxx} + 3abvv_{xx} + 3auv_x - w_x = 0 \tag{22}$$

$$u_t + abu_{xxx} + 3auu_x - bw_{xx} + bv(3au_x - 2w)_x = 0 \tag{23}$$

where the first equation reminds us of the Miura transformation [7, 8]. The third equation obviously decouples from the others if we choose

$$w_x = \tfrac{3}{2}au_{xx}. \tag{24}$$

However, taking (21) into account, one finds a more general solution of the decoupling problem, namely

$$w_x = \tfrac{3}{2}au_{xx} + cv_x \tag{25}$$

with a constant, c, (23) then becomes

$$u_t - cu_x + 3auu_x - \tfrac{1}{2}abu_{xxx} = 0 \tag{26}$$

which for

$$a = -2 \qquad b = 1 \qquad c = 0 \tag{27}$$

is the KdV equation as given in [8], for example. We observe that the parameter c simply reflects the effect of a special Galilean transformation‡. To summarize, the zero curvature equation together with the restriction (25) on the connection A leads to the KdV equation.

With the help of (21), the equation (22) is turned into

$$v_t - cv_x - \tfrac{1}{2}abv_{xxx} + 3abv^2v_x = 0 \tag{28}$$

from which we recover what is known as a mKdV equation [8]. It is surprising that both, the KdV and the mKdV equation appear jointly in our mathematical framework.

In the above differential calculus it is consistent to impose the additional condition that the 1-form ξ is closed, i.e. $d\xi = 0$. The above formulae remain valid, except that now

$$dx\,dx = 0. \tag{29}$$

The zero curvature condition is then slightly less restrictive. It still leads to (22) and (23), but (21) is replaced by the weaker equation

$$\frac{1}{b}u = v_x + v^2 + \lambda \tag{30}$$

with a function $\lambda(t)$. For constant λ we rediscover what is sometimes referred to as the 'Miura–Gardner transformation' (see [9], for example).

From the gauge transformation rule (2) we obtain

$$d\psi = \psi A - A'\psi \tag{31}$$

† We still have the freedom to modify the differential calculus at the level of 2-form relations, i.e. to modify the generalized wedge product by setting certain products of differentials, like $dx\,dx$, to zero. The corresponding terms in (20) then simply drop out. In general it is more natural to proceed without such extra conditions. There is some motivation, however, to impose on the 1-form ξ the condition $d\xi = 0$, see the following section.

‡ For the Galilean transformation $x' = x + ct$, $t' = t$ we have $\partial_{t'} = \partial_t - c\partial_x$, $\partial_{x'} = \partial_x$.

where ψ is an invertible (smooth) function. Using (13) and (19) this becomes

$$\psi_t + ab\psi_{xxx} = (w - w')\psi - 3au'\psi_x - 3abv'\psi_{xx} \tag{32}$$

$$b\psi_{xx} = (u - u')\psi - 2bv'\psi_x \tag{33}$$

$$\psi_x = (v - v')\psi. \tag{34}$$

If $\mathrm{d}\xi = 0$, a simple solution of the zero curvature condition and (25) is given by $A' = \lambda\xi$ with a constant λ. This determines a trivial solution of the KdV equation. The above equations then take the form

$$\psi_t + ab\psi_{xxx} = w\psi - 3a\lambda\psi_x \tag{35}$$

$$b\psi_{xx} = (u - \lambda)\psi \tag{36}$$

$$\psi_x = v\psi \tag{37}$$

and enforce that A also has vanishing curvature. Restricting the gauge transformation further in such a way that A satisfies (25) and thus determines a solution of the KdV equation, and making use of the last of the above equations, the first equation becomes

$$\psi_t + ab\psi_{xxx} = \left(\frac{3a}{2}u_x + f\right)\psi + (c - 3a\lambda)\psi_x \tag{38}$$

where an arbitrary function $f(t)$ arose from integration of (25). Using the Sturm–Liouville equation (36), (38) can be written as

$$\psi_t = \left(\frac{a}{2}u_x + f\right)\psi + (c - au - 2a\lambda)\psi_x. \tag{39}$$

For $a = -2$, $c = 0$ and special choices of f this is the time-evolution equation for eigenfunctions of the Schrödinger operator associated with the KdV equation (cf [8], p 101).

4. The KP equation

In the differential calculus introduced in the previous section it is tempting to replace the 1-form ξ by the differential $\mathrm{d}y$ of a third coordinate function y. For functions f which do not depend on y we recover the formulae of the previous section. But the extension to functions of t, x, y requires nontrivial modifications. A consistent differential calculus on $\mathcal{A} = C^\infty(\mathbb{R}^3)$ is obtained by supplementing the relations (17) with†

$$[\mathrm{d}t, y] = [\mathrm{d}y, y] = 0 \qquad [\mathrm{d}x, y] = 3a\,\mathrm{d}t. \tag{40}$$

Then

$$\mathrm{d}t\, f = f\,\mathrm{d}t \qquad \mathrm{d}y\, f = f\,\mathrm{d}y + 3af_x\,\mathrm{d}t \qquad \mathrm{d}x\, f = f\,\mathrm{d}x + 2bf_x\,\mathrm{d}y + 3a(f_y + bf_{xx})\,\mathrm{d}t \tag{41}$$

and

$$\mathrm{d}f = (f_t + 3af_{xy} + abf_{xxx})\,\mathrm{d}t + (f_y + bf_{xx})\,\mathrm{d}y + f_x\,\mathrm{d}x. \tag{42}$$

The set of 2-form relations (18) is extended by

$$\mathrm{d}t\,\mathrm{d}y + \mathrm{d}y\,\mathrm{d}t = \mathrm{d}x\,\mathrm{d}y + \mathrm{d}y\,\mathrm{d}x = \mathrm{d}x\,\mathrm{d}x = 0 \tag{43}$$

(and modified via $\xi = \mathrm{d}y$, of course). Now $\mathrm{d}t\,\mathrm{d}x$, $\mathrm{d}t\,\mathrm{d}y$, $\mathrm{d}y\,\mathrm{d}x$ is a basis of the space of 2-forms as a left $\mathcal{A}$-module.

† This is really the minimal extension of (17) obtained via $\xi \mapsto \mathrm{d}y$. Note that $[\mathrm{d}x, y] = [\mathrm{d}y, x]$ using the Leibniz rule and $[x, y] = 0$.

Any 1-form A can be expressed as

$$A = w\,\mathrm{d}t + u\,\mathrm{d}y + v\,\mathrm{d}x \tag{44}$$

with function u, v, w. Regarded as a $GL(1,\mathbb{R})$ connection 1-form, the curvature is

$$\begin{aligned} F = \mathrm{d}A + AA = \{&-w_y - bw_{xx} + u_t + 3au_{xy} + abu_{xxx} \\ &+3auu_x - 2bvw_x + 3av(u_y + bu_{xx})\}\,\mathrm{d}t\,\mathrm{d}y \\ &+\{-w_x + v_t + 3av_{xy} + abv_{xxx} + 3auv_x + 3av(v_y + bv_{xx})\}\,\mathrm{d}t\,\mathrm{d}x \\ &+\{-u_x + v_y + bv_{xx} + 2bvv_x\}\,\mathrm{d}y\,\mathrm{d}x. \end{aligned} \tag{45}$$

This vanishes iff

$$u_x = v_y + b(v_x + v^2)_x \tag{46}$$

$$w_x = v_t + 3av_{xy} + abv_{xxx} + 3auv_x + 3av(v_y + bv_{xx}) \tag{47}$$

$$w_y + bw_{xx} = u_t + 3au_{xy} + abu_{xxx} + 3auu_x - v(2bw_x - 3a(u_y + bu_{xx})). \tag{48}$$

The next step parallels that of the KdV case treated in the previous section. v is obviously eliminated from the last equation by setting

$$w_x = \frac{3a}{2b}u_y + \frac{3a}{2}u_{xx}. \tag{49}$$

Motivated by the KdV example, we shall consider the more general expression

$$w_x = \frac{3a}{2b}u_y + \frac{3a}{2}u_{xx} + cv_x \tag{50}$$

where c is an arbitrary constant. Taking (46) into account, (48) then reduces to

$$w_y = u_t - c(u_x - v_y) + \frac{3a}{2}u_{xy} - \frac{ab}{2}u_{xxx} + 3auu_x. \tag{51}$$

Now there is an integrability condition. Comparing the results obtained by differentiating (50) with respect to y and (51) with respect to x, we obtain

$$\left(u_t - cu_x - \frac{ab}{2}u_{xxx} + 3auu_x\right)_x - \frac{3a}{2b}u_{yy} = 0 \tag{52}$$

which is the KP equation (for the choices (27) of the constants a, b, c, see [8] for example). Again, via a Galilean transformation the constant c can be eliminated. Though on the level of the zero curvature equations our ansatz (50) with $c \neq 0$ does not really decouple the variables because of the term cv_y, the latter does not enter the integrability condition.

Let us now turn to the equation for v which resulted from the zero curvature condition. Taking (50) into account, we have

$$\frac{3a}{2b}u_y = v_t - cv_x + \frac{3a}{2}v_{xy} - \frac{ab}{2}v_{xxx} - 3abv_x^2 + 3avv_y + 3auv_x. \tag{53}$$

Expressing v as

$$v = q_x \tag{54}$$

with a function q, (46) becomes

$$u_x = q_{xy} + b(q_{xx} + q_x^2)_x \tag{55}$$

and thus

$$u = q_y + b(q_{xx} + q_x^2) + f \tag{56}$$

where f is a function which does not depend on x, i.e. $f(t, y)$. Now we can eliminate u from (53) and obtain

$$\left(q_t - cq_x - \frac{ab}{2}q_{xxx} + abq_x^3\right)_x + 3a(q_y + f)q_{xx} - \frac{3a}{2b}(q_{yy} + f_y) = 0. \quad (57)$$

Expressing f as $f = h_y$ with a function $h(t, y)$, a field redefinition $q \mapsto q - h$ eliminates f from the last equation and we get

$$\left(q_t - cq_x - \frac{ab}{2}q_{xxx} + abq_x^3\right)_x + 3aq_y q_{xx} - \frac{3a}{2b}q_{yy} = 0. \quad (58)$$

This equation may be called a 'modified KP equation' (mKP). We note that

$$\mathrm{KP} = (\partial_y + b\partial_x^2 + 2bq_x\partial_x + 2bq_{xx})\mathrm{mKP}. \quad (59)$$

Hence, given a solution q of the mKP equation, then u determined by (56) is a solution of the KP equation.

5. The Boussinesq equation

Restricting the KP equation (52) to the hypersurface $t = 0$ and renaming y as t afterwards, we arrive at the equation

$$u_{tt} + \frac{2bc}{3a}u_{xx} - b(u^2)_{xx} + \frac{b^2}{3}u_{xxxx} = 0 \quad (60)$$

which includes the Boussinesq equation (see [8], for example). The differential calculus associated with this equation is obtained as a reduction of the calculus which led us to the KP equation in the last section. First, we have to replace dt by some 'abstract' 1-form ξ. Then, keeping our renaming $y \mapsto t$ in mind, the commutation relations defining the differential calculus of the previous section yield

$$\begin{aligned} &[\mathrm{d}t, t] = [\xi, t] = [\xi, x] = 0 \\ &[\mathrm{d}t, x] = [\mathrm{d}x, t] = 3a\xi \qquad [\mathrm{d}x, x] = 2b\,\mathrm{d}t. \end{aligned} \quad (61)$$

The differential of a function f is now given by

$$\mathrm{d}f = (3af_{tx} + abf_{xxx})\xi + (f_t + bf_{xx})\,\mathrm{d}t + f_x\,\mathrm{d}x. \quad (62)$$

Of course, we could have started with the differential calculus determined by these relations and derived the Boussinesq equation in the same way as we derived the KdV equation in section 3. In this case we in fact need to add a term proportional to v_x in the decoupling ansatz for w_x in order to recover the u_{xx} part of (60). The term was of minor importance in our previous examples (see (25) and (50)).

6. Conclusions

Crucially underlying this work is the observation that with respect to a noncommutative differential calculus already the field strength (curvature) of a single connection 1-form (i.e. a $GL(1, \mathbb{R})$ or a $U(1)$ connection) involves nonlinear terms. With the ordinary calculus of differential forms on a manifold, a matrix of connection 1-forms is needed to achieve that. This observation suggested to investigate which well known nonlinear field equations, and in particular soliton equations, can be formulated as zero curvature conditions for a single connection 1-form with respect to a suitable noncommutative differential calculus. We found that the Burgers, KdV, KP and Boussinesq equation indeed admit such a formulation.

Of most interest is the fact that the differential calculus associated with the KdV equation has a natural extension and, following the steps which led to the KdV equation, we are led straight to the KP equation.

Nevertheless, though things fit surprisingly well together, a deeper understanding why this is so is still lacking. For a certain class of completely integrable models noncommutative differential calculus has indeed led to a rather complete understanding and a recipe to construct new integrable models [10, 11]. There is therefore much hope to achieve a comparable understanding of the structures presented in this work.

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